Statistical properties of chaos at onset of electroconvection in a homeotropically aligned nematic layer

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The recently discovered chaos at the onset of electroconvection in a homeotropically aligned nematic layer caused by slow random long-wavelength modulations of a roll pattern is discussed. The temporal autocorrelation function for components of the order parameter is expressed in terms of probability density for random drift velocity of the pattern. It is shown that despite the fact that the problem has at least two different characteristic times associated with the slow pattern dynamics, only one of them enters into the autocorrelation function. [S1063-651X(99)12003-8]

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In the present paper a ''paradox of two correlation times'' (see below) for spatiotemporal chaos at the onset of electroconvection in a homeotropically aligned nematic layer (the director is perpendicular to confining surfaces) is discussed, exploded and removed. As a result, a simple formula for the temporal autocorrelation function of the problem is obtained.

Systematic experimental study of disordered patterns in electroconvection under the specified conditions began rather recently and was conducted, generally, by two groups; see Refs. [1–3] and [4–6]. Both groups observe the disordered patterns at the very onset of electroconvection. However, there is a certain discrepancy in details of their observations and measurements. The discrepancy probably is attributable to differences in cells and material constants. To avoid misunderstanding, we emphasize that in the case of discrepancy we refer to the results of Refs. [1–3] rather than to those of Refs. [4–6]. According to these results, spatiotemporal chaos (STC) is caused by continuous random spatiotemporal modulation of the local orientation of convective rolls. The characteristic spatial scale of the modulation Λ (macroscale) diverges at $\epsilon=0$, where $\epsilon\equiv(V^2-V_c^2)/V_c^2$ stands for the normalized control parameter, V for the applied voltage and V_c for the threshold of convective instability. At $0<\epsilon\ll1$ this scale is much greater than the corresponding microscale of the problem λ (double the roll diameter). It makes the problem qualitatively different from other examples of chaos at onset, such as, e.g., STC associated with Hopf bifurcation supplemented by the Benjamin-Feir destabilization, or the one caused by Küppers-Lortz instability (see, for instance, review [7], or for more recent results Refs. [8]), where the destabilizing modes have the same spatial scale as that for the pattern itself.

The phenomenon is observed in a nematic with negative dielectric anisotropy, where the Fréedericksz transition [9] precedes the convective instability. Due to the transition the purely homeotropic alignment of the director is distorted, so that at the threshold of the convective instability the director has a nonzero projection on the plane of the nematic layer. Since the azimuthal orientation of the director is not imposed by any external factor the system is degenerate with respect to arbitrary rotations of the director around any axis perpendicular to the layer's plane. For this reason, beyond the threshold of convective instability the corresponding long-wavelength rotational modes become undamped [10]. On the other hand, in a patterned state beyond V_c the local orientation of the roll wave vector $\mathbf{k}$ is rigidly connected with the azimuthal orientation of the director (see, e.g., [7]). Thus, any change of the former causes the corresponding change of the latter. For rolls that are perfectly straight and parallel to each other, a change in azimuthal orientation of $\mathbf{k}$ means rotation of the pattern as a whole (Goldstone mode). In this case, in a weakly nonlinear regime ($0<\epsilon\ll1$), undamped modes detaching from the Goldstone one should correspond to random rotations of different fragments of the pattern with the characteristic spatial scale of the fragment of order Λ. Thus, changes of the components of the order parameter (such as charge density, the velocity vector, etc.) at a given fixed point are caused by drift of the pattern by this point associated with the rotations.

One of the most important characteristics of any random process is its autocorrelation function. For the problem in question, temporal autocorrelation functions have been studied already, based upon the results of computer simulations [11] and real experiment [2,3,6]. Both the computer simulations and the experiments say that the problem has a single correlation time τ scaled as $1/\epsilon$ and that the autocorrelation functions fall off to zero monotonically with increase of time.

Note, however, that according to the described nature of the STC, the problem has at least *two* different characteristic time scales, viz., λ/v_0 and Λ/v_0, where v_0 stands for the characteristic drift velocity (provided there is a unique v_0; otherwise the case becomes even more complicated). If a pattern consists of strictly spatially periodic rolls, its rotation as a whole gives rise to an autocorrelation function periodic in time with the period λ/v_0 [12]. For this reason, it seems natural to expect that for the true patterns distorted by chaotic long-wavelength modulations, the autocorrelation functions are generally oscillating. The oscillation period should

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be of order λ/v_0, while the oscillation amplitude should decay on a much bigger time scale of order Λ/v_0. In other words, there should be *two* substantially different correlation time scales for the oscillations themselves and their decay, respectively. This expectation disagrees drastically with results of Refs. [2,3,6,11] (the paradox of ‘‘two correlation times’’). One of the goals of the present paper is to show that the expectation is wrong and the above-mentioned numerical and experimental results are explained in a quite simple manner.

Let $u(\mathbf{r},t)$ be a scalar quantity denoting any component of the order parameter. Since we consider the autocorrelation function at a given fixed value of $\mathbf{r}$, only the time dependence of u is important. According to the specified scenario, such a dependence originates in a drift of almost spatially periodic roll pattern, so it has the form

$$u = A\sin\left(\int_0^t \omega(t')dt' + \alpha_0\right),$$

where A may be regarded as a constant [13], $\omega \equiv kv$, k is a constant roll's wave number, v is the drift velocity slowly varying in time, and α_0 is a constant initial phase. Then, for the autocorrelation function $\phi(t)$ we have

$$\begin{aligned}\phi(t) &= \langle u(\mathbf{r},t'+t)u(\mathbf{r},t')\rangle \\ &= A^2\left\langle \sin\left(\int_0^{t'+t}\omega(t'')dt''+\alpha_0\right)\right. \\ &\quad \left.\times \sin\left(\int_0^{t'}\omega(t'')dt''+\alpha_0\right)\right\rangle \\ &\simeq \frac{A^2}{2}\left\langle \cos\left(\int_{t'}^{t'+t}\omega(t'')dt''\right)\right\rangle \\ &\simeq \frac{A^2}{2}\left\langle \cos\left[\omega(t')t+\frac{1}{2}\dot{\omega}(t')t^2+\cdots\right]\right\rangle, \end{aligned} \qquad (1)$$

here $\dot{\omega}$ denotes time derivative and

$$\langle \cdots \rangle \equiv \lim_{T'\to\infty} \frac{1}{T'}\int_0^{T'} (\cdots)dt'$$

(we employed trivial trigonometric transformations and took into account that the cosine of a sum of two phases is a rapidly oscillating function of t' and therefore its average $\langle\ \rangle$ vanishes).

Now note that at ϵ small enough and *any* fixed t the characteristic value of $\dot{\omega}t^2$ in the argument of the cosine in Eq. (1) is always much smaller than one, i.e., $\dot{\omega}t^2$, and that all terms that are higher order in t are also negligible quantities. Actually, a noticeable change of ω requires a time of order Λ/v_0. It yields the estimation $\dot{\omega}t^2 = O(kv_0^2t^2/\Lambda)\to 0$ at $\epsilon\to 0$ due to the vanishing of v and the divergence of Λ, in this limit. At finite small ϵ neglect of $\dot{\omega}t^2$ results in the following restriction for t:

$$t \ll \frac{1}{v_0}\sqrt{\frac{\Lambda}{k}}. \qquad (2)$$

On the other hand, we are interested in the case in which the phase of the cosine in Eq. (1) may be big enough. It results in the condition $\omega_0 t \gg 1$, where $\omega_0 \equiv kv_0$. Consistency of this condition with Eq. (2) gives rise to the inequality $\sqrt{\Lambda k}\gg 1$, which is always satisfied at small ϵ. Thus, at small ϵ the leading approximation to the autocorrelation function reads

$$\phi(t) = \frac{A^2}{2}\langle \cos[\omega(t')t]\rangle + \cdots, \qquad (3)$$

so that the second characteristic time of the problem Λ/v_0 does not enter into the expression at all.

Note that the averaging over t' is equivalent to that over ω. Actually, transforming the integral over t' into an integral sum by discretizing of ω [letting $\omega(t')$ have only discrete values from a set ω_n, where the ‘‘spacing’’ $\Delta\omega \equiv \omega_{n+1} - \omega_n$ has the same value at any n], and rearranging terms in the sum, putting together all terms with the same value of ω_n, we arrive at the expression

$$\frac{1}{T'}\int_0^{T'}\cos[\omega(t')]dt' = \lim_{\Delta\omega\to 0}\sum_n (\cos\omega_n) \times \frac{1}{T'}(\Delta t_n^{(1)\prime}+\Delta t_n^{(2)\prime}+\cdots), \qquad (4)$$

where $\Delta t_n^{(1)\prime}, \Delta t_n^{(2)\prime}, \ldots$ is the entire set of time segments within which the discretized function $\omega_n(t)$ has a given value. Bearing in mind that by definition

$$\lim_{T'\to\infty}\frac{1}{T'}(\Delta t_n'^{(1)}+\Delta t_n'^{(2)}+\cdots)$$

is the probability for a random function $\omega_n(t)$ to have a given value, setting $\Delta\omega$ to zero, and replacing the sum in Eq. (4) by the corresponding integral, we transform Eq. (3) into the form

$$\phi(t) = \frac{A^2}{2}\int (\cos\omega t)p(\omega)d\omega, \qquad (5)$$

where $p(\omega)$ stands for the probability density. From the normalization condition, it follows that $\int p(\omega)d\omega = 1$. Thus, integral (5) always converges, being a smooth, differentiable function of t. Note that $p(\omega)$ should be an even function of ω due to the equivalence of clockwise and counterclockwise rotations, so that $\langle\omega^{2n+1}\rangle = 0, \quad n = 0,1,2,\ldots$.

Equation (5) yields a number of important conclusions. First, expression (5) is an even function of t, which has a smooth maximum at $t=0$ and tends to zero at $t\to\infty$. In other words, none of the generic properties of autocorrelation functions [14] is violated.

Second, none of the correlation characteristics of random ω enters into Eq. (5). The only quantity the correlation function $\phi(t)$ depends on is the probability density $p(\omega)$.

Third, if $p(\omega) = \delta(\omega-\omega_0)$, which corresponds to a strictly spatially periodic pattern with a fixed drift velocity, Eq. (5) yields $\phi(t) = (A^2/2)\cos\omega_0 t$, i.e., the mentioned oscillating behavior of the autocorrelation function is recovered. In the opposite limit, when $p(\omega)$ is a smooth function with a certain unique characteristic ω_0 (now ω_0 describes the dispersion of the distribution), integration in Eq. (5) results in a $\phi(t)$ of the form $(A^2/2)f(\omega_0 t)$, where $f(x)$ is a smooth function, satisfying the condition $f(0)=1$ and falling off to zero at x of order one. It means the problem possesses a single correlation time, which is nothing but $\tau=1/\omega_0$.

It should be emphasized that the obtained results are valid at $t \gg \tau$, i.e., the long-time behavior of the autocorrelation function is discussed. Only the very tail of $\phi(t)$ at t violating condition (2) deviates from expression (5). This tail, however, is not of interest, since $\phi(t)$ is practically zero in this region.

To understand which of the limits, $p(\omega)=\delta(\omega-\omega_0)$ or a smooth broad distribution, is closer to reality, we have to remember the proposed mechanism of chaotization associated with rotations of different fragments of a roll pattern around random axes with random angular velocities. Obviously, in this case the distribution $p(\omega)$ must have a certain nonzero dispersion. Note further that v_0 may be estimated as a product of the characteristic angular velocity of the rotation ($\partial\varphi/\partial t$, where φ is the azimuthal angle of the director) and the characteristic spatial scale of the rotating fragment, which is Λ. Adopting for these quantities the ϵ scaling employed in Refs. [11], i.e., supposing that $\Lambda \propto 1/\sqrt{\epsilon}$ and rotation through an angle of $O(\sqrt{\epsilon})$ takes time $\propto 1/\epsilon$, we obtain a *unique* characteristic value $\omega_0 \propto \epsilon$. Thus, the model in question gives rise to a smooth broad distribution $p(\omega)$ with a *single* characteristic frequency. In turn, it yields rather a monotonic autocorrelation function with a single correlation time $\tau \propto 1/\epsilon$, which agrees with the mentioned numerical and experimental results.

Let us consider a particular example of the Gaussian distribution of random ω, namely,

$$p(\omega)=\frac{1}{\omega_0\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^2\right]. \tag{6}$$

In this case the integral in Eq. (5) may be evaluated in terms of hypergeometric functions; however, it is more convenient to expand the cosine function in Taylor series and to integrate it term by term. It gives rise to the expression

$$\phi(t)=\frac{A^2}{2}f(\omega_0 t),$$
$$f(x)\equiv\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!!}. \tag{7}$$

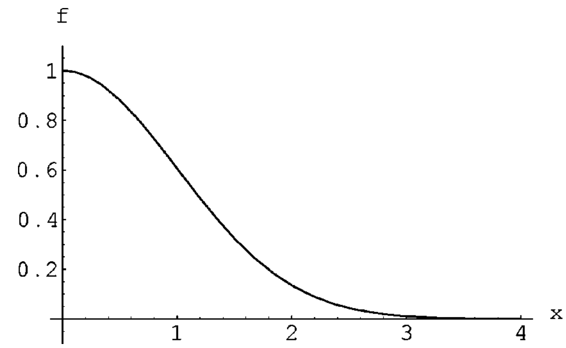

FIG. 1. The autocorrelation function $f(x)$, obtained for the Gaussian distribution of random ω [see Eqs. (6), (7)].

Series (7) converges at any finite x. The function $f(x)$ defined according to Eq. (7) falls off monotonically to zero with increase of x; see Fig. 1. Despite the fact that $f(x)$ is not reduced to a simple exponential function, its approximation by such a function may be quite reasonable.

Basically, Fig. 1 is in good agreement with experimental curves presented in Refs. [2,3]. A certain difference in details (broader maximum at $x=0$ and sharper decay at $x \ll 1$ compared to the experimental curves) should not be overevaluated. First, our intention is to understand the general features of the phenomenon within the framework of the simplest model. As a result, a number of details of the problem that might affect the behavior of the true autocorrelation function are not taken into account. Second, we do not have any specific reason to suppose that the true $p(\omega)$ in real experiments does have the Gaussian form. Such a form is taken just as an example to illustrate a more general discussion. Note also that the small parameter of the developed approach is the ratio λ/Λ. In the experiments referenced, this ratio is about 5–7 (see snapshots of disordered patterns in Refs. [2,3]), i.e., it can hardly be regarded to be small enough for quantitative comparison of the theory and the experiments.

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[13] It means we neglect defect generation, since A must vanish in the defect cores. The neglect would be erroneous for the planar alignment, when orientation of convective rolls is fixed by boundary conditions and the defect generation is an important mechanism for transition to chaos (the so-called defect-mediated turbulence) [7]. On the contrary, the problem under consideration has an additional degree of freedom to make the pattern chaotic, viz., the mentioned undamped rotations of its different fragments with respect to each other. Regarding the defect generation, for point defects it is suppressed due to smallness of ϵ [2,3]. As for dislocation lines, which may arise at boundaries of the different pattern's fragments, their quantitative measure is the ratio λ/Λ, which vanishes at ϵ tending to zero. Thus, both the point and the line defects should not affect statistical properties of the pattern at small ϵ.

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